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The search for higher symmetry in string theory

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Some remarks are made about the nature and role of the search for higher symmetry in string theory. These symmetries are most likely to be uncovered in a mysterious 'unbroken phase', for which (2+1)-dimensional gravity provides an interesting and soluble model. New insights about conformal field theory, in which one gets 'out of flatland' to see a wider symmetry from a higher-dimensional vantage point, may offer clues to the unbroken phase of string theory.

1. Search for the unbroken phase

I thought I might begin by stating my general outlook on string theory.

In my opinion, the basic challenge in string theory is not, as sometimes said, to 'understand non-perturbative processes in string theory'. In fact, the basic problem does not lie in the quantum domain at all. The basic problem, now and probably for many years to come, is to understand the classical theory properly.

Understanding the classical theory means above all understanding the geometrical ideas that parallel those of general relativity. It is indeed a key aspect of the fascination of string theory that in string theory, general relativity is modified *even in the classical domain*. The key geometrical ideas upon which string theory should be founded must be worthy successors to the principle of equivalence and riemannian geometry, and they must lead to what we now know of as string theory as ineluctably as the principle of equivalence leads to general relativity.

Not only is the search for geometric understanding crucial, it is the key area where progress can be expected in the next stage of development. The advance in viewpoint that we need is probably big, but perhaps no bigger than the advance made long ago when 'dual models' were reinterpreted as 'string theories'. I suppose my basic outlook on physics is that conceptual problems such as those cited above have answers, and these answers are eventually found by human beings. One might not say the same for dynamical problems.

So the central task is to

(i) find the right degrees of freedom;
(ii) find the right invariance group; and
(iii) formulate the right lagrangian.

These are, of course, the usual questions of string field theory, and I consider them to be the right questions though string field theory as we know it is very likely not based on the right degrees of freedom.

In thinking about these matters, it is sobering to realize how little we know with certainty. Thus it is often said that 'string theory is a theory of extended objects', and Riemann surfaces are often felt to play a crucial role. But the $1/N$ expansion of quantum chromodynamics (QCD) may offer a cautionary tale. As we have long known, QCD with $SU(N)$ gauge group has an expansion formally very similar to the topological expansion of string theory, with mesons and glueballs playing the role of open and closed strings, and a mesh of planar gluons playing the role of the string world sheet. In this picture, it is confinement of colour that creates effective 'Riemann surfaces'. We do not know of a simple Veneziano-like description of the $1/N$ expansion of QCD, but one may exist. (And if there is an area of QCD theory where a real breakthrough can be made, this is it. Certainly, the search for such a description has been one of the main motivations of the Soviet school of string theorists, throughout the 1980s.)

QCD from this viewpoint looks like a 'theory of extended objects', all based on Riemann surfaces. Yet conceptual understanding from this point of view is probably hopeless. It is necessary to learn that the correct degrees of freedom are quarks and gluons, before learning that the correct notion is gauge theory.

At least in thought experiments, one can neatly see the world of extended objects and Riemann surfaces come to an end in QCD, by heating to the deconfinement temperature, T_{dec}. Since the early work on the deconfinement transition (Polyakov 1978; Susskind 1979) it has been suspected that a similar phenomenon may be occurring in string theory at the so-called Hagedorn temperature, T_{H}. There are a variety of reasons to suspect that this may be so. One argument that I find fairly convincing is that in string theory, the free energy begins in genus one if $T \leqslant T_{\mathrm{H}}$, but receives a genus zero contribution if $T \geqslant T_{\mathrm{H}}$ (Atick & Witten 1988). This parallels an analogous behaviour in QCD.

Another reason to suspect the occurrence in string theory of an analogue of the deconfinement transition is that, on the face of it, the building blocks that we are currently using in our attempts to describe string theory do not seem adequate for the job. String fields are too messy, and two-dimensional field theory is too singular. Let me pause for a word on the latter point. We recall that unitary conformal field theories come in finite-parameter families (the number of parameters being the number of relevant operators). This is a very different sort of thing from what we want to describe time-dependent processes, where it should be possible to excite arbitrary massive modes of the string. I believe that this is a crucial issue, and in no way a technicality. In any serious attempt to study lorentzian signature sigma models with time-dependent fields, the singularities of quantum field theory will appear with a vengeance, with operators of arbitrarily negative dimension appearing on the scene. It is significant that this subject has been so little investigated in print.

General relativity should again offer the paradigm of what we want. The basic concept of a metric is flexible and elementary. Choosing a metric does not require solving any equations. We should expect to one day develop string theory with tools as flexible and elementary as the concept of a riemannian metric. This may be possible only by understanding the 'unbroken phase' of string theory, with the higher symmetries restored. What properties do we expect of this phase?

There are many reasons to believe that in string theory there is no such thing as distances less than the fundamental length $\sqrt{\alpha'}$. Among the reasons for believing this are the duality in distance between radius R and radius α'/R, the fact that (as in the paper by David Gross, this Symposium) one cannot probe distances below $\sqrt{\alpha'}$ by using high-energy, fixed-angle scattering, and the fact that the growth of the free energy above the Hagedorn temperature seems to be slower than in field theory. What one might imagine would be a world in which at distances above $\sqrt{\alpha'}$, normality prevails, but at distances below $\sqrt{\alpha'}$, not just physics as we know it but local physics altogether has disappeared. There will be no distance, no times, no

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energies, no particles, no local signals – only differential topology, or its string theoretic successor.

To clarify this a bit, let us note that one often thinks of four-dimensional general relativity, in the phase we live in, as a system in which the space-time symmetries are unbroken. On reflection, though, one sees that the non-zero expectation value of the metric, $g_{\mu\nu} \sim \eta_{\mu\nu}$, violates *local* general covariance and leaves only the *global* Poincaré symmetries. To restore local general covariance, one would need $\langle g_{\mu\nu}\rangle = 0$. But in the absence of an expectation value of the metric, there is no notion of distance, time or energy, and there is no way to have particles or local signals because there is no way for these to stay inside the light cone.

In general relativity, we are usually satisfied with unbroken global symmetry. We do not usually hunger to restore the local symmetry and thus bring physics to an end. In string theory, however, the graviton is just one of a tower of states. Our crucial task is to understand the symmetries associated with the massive states, the masses presumably arising by a Higgs-like mechanism. In string theory, an expectation value of the metric breaks the higher symmetries even as global symmetries. To restore the higher symmetries of the string even as global symmetries, one needs $\langle g_{\mu\nu}\rangle = 0$, and in this case general covariance, and the whole higher invariance group of which it is part, is restored as a *local* symmetry. As we have already noted, restoration of general covariance makes local physics impossible, and this is why I believe that string theory must be understood in a framework in which locally there is nothing there.

The most concrete picture I can offer of such local symmetry restoration comes in (2+1)-dimensional gravity. Before entering into this, let us ask why (3+1)-dimensional gravity is unrenormalizable. I write the lagrangian in vierbein formalism:

$$\mathscr{L} = \int e^a \wedge e^b \wedge (\mathrm{d}\omega + \omega \wedge \omega)^{cd} \cdot \epsilon_{abcd}. \tag{1.1}$$

(Here e^a is the vierbein, ω^{ab} the spin connection, and ϵ_{abcd} the alternating symbol with $\epsilon_{0123} = +1$.) Note that if we regard e and ω as fields of dimension one (this is conventional for ω, but e is usually regarded as a field of dimension zero), then every term in (1.1) has dimension four, so we might think that general relativity is renormalizable in four dimensions. This is not so; where is the mistake?

The error is that if e and ω have positive dimension, the short distance behaviour must be governed by $e = \omega = 0$. This is what I mean, in general relativity, by the 'unbroken phase' with *local* general covariance restored. But we do not understand four-dimensional general relativity in the 'unbroken phase'. Mathematically, we cannot expand (1.1) around $e = \omega = 0$ because there is no quadratic term. Physically, gravitons could not propagate in such a region, so in four dimensions the interface between the broken and unbroken phases must be quite subtle.

In 2+1 dimensions, the story is different. Once initial reluctance is overcome, the action,

$$\mathscr{L} = \int (e^a \wedge (\mathrm{d}\omega + \omega \wedge \omega)^{bc} + \lambda e^a \wedge e^b \wedge e^c)\, \epsilon_{abc} \tag{1.2}$$

(I have incorporated a cosmological constant term), can readily be expanded around $e = \omega = 0$. What is more, the resulting expansion (if placed in the framework of gauge theory, as is made possible by observations in (Achúcarro & Townsend 1986; Witten 1989*a*) is renormalizable by power counting, and is easily seen to be finite. Given that (1.2) makes sense and is finite in the unbroken phase, it can be expanded around any classical solution without spoiling the finiteness.

Macroscopically, e and ω can be given the usual riemannian interpretation of general relativity. Microscopically, this is not so. After all, the short-distance régime is an expansion around $e = 0$, and a space-time point with $e = 0$ (or even $\det e = 0$) is a singularity from the standpoint of riemannian geometry. In a gauge theory interpretation, we do not regard such occurrences as singularities, and this is a crucial departure from the riemannian interpretation. Actually, in the gauge theory interpretation, e and ω can be gauged away locally, so in a very precise sense there is 'nothing' locally. Nevertheless, in the large, one has macroscopic space-times. (These macroscopic space-times are empty, and have only global dynamics, because there are no gravitons in $2+1$ dimensions. It is this that makes $(2+1)$-dimensional gravity simple, while the incorporation of the unbroken phase will be vastly more difficult in $3+1$ dimensions.)

I would like to believe that something like this occurs in nature, but far more subtle, because there are signals propagating in the 'broken phase' and the interface between them must be at least as subtle as in QCD.

In seeking such an understanding, we are in the situation of the proverbial man who has lost his key and is looking for it in the dark. There is little to do except to look under a well-lit lamp-post and hope for the best. But it is up to us to pick a promising lamp-post to look under. The missing key is the geometrical key to string theory. My own favourite lamp-post – about which I will say a few words in the next section – is the investigation of geometrically interesting structures associated with quantum field theory, in dimensions ranging from two (the world-sheet) to four (the dimension of space-time, at least macroscopically). These are the most interesting dimensions in geometry, topology and physics. The idea that to understand two-dimensional field theory one must get out of 'flatland' and find a higher-dimensional perspective was originally advocated by Sir Michael Atiyah; I am grateful for the inspiration he has provided.

2. Higher symmetries in conformal field theory

Let us first recall that in *four* space-time dimensions, the *spin* of a massless particle is a representation of SO(2) and so is an integer or half-integer. Thus fermions, gauge bosons and gravitons may have *positive* or *negative* helicity. The chiral decomposition for fermions is one of the central facts in particle physics, but this is much less true for bosons, though the polarization of light is certainly one of the basic observations in electromagnetism, and in QCD the self-dual Yang–Mills equations,

$$F_{\mu\nu} = \tilde{F}_{\mu\nu}, \tag{2.1}$$

do enter. The chiral decomposition is less important for bosons because it cannot be made locally.

Twistor theory (for introductory articles see Lerner & Sommers (1978)), in which four-dimensional space-time R^4 is replaced by a higher-dimensional space CP^3, exhibits a remarkable higher symmetry of the chiral or self-dual theory. All information is encoded globally. The challenge in twistor theory has always been to combine the left-handed and right-handed theories. (A proposal was made in Witten (1978) and Isenberg *et al.* (1978). It remains to be seen if this proposal can offer any guidance for conformal field theory.) This is a difficult problem, because the Einstein and Yang–Mills equations are not 'rational' in the language of conformal field theory.

The chiral decomposition in four dimensions has, of course, an analogue in two dimensions in the decomposition in left-movers and right-movers. This decomposition is powerful primarily in the so-called rational conformal field theories. Its use in that context is practically the only truly two-dimensional tool we know of in conformal field theory. When this tool is not available, we know scarcely more about conformal field theory in two dimensions than in four dimensions.

One way that the chiral decomposition enters in conformal field theory is this. Let us work on a Riemann surface Σ of genus g, with the moduli denoted as m^i. A conformal field theory is said to be 'rational' if its partition function $Z(m^i, \bar{m}^j)$ has an expansion,

$$Z(m^i, \bar{m}^j) = \sum_{\lambda=1}^{k} f_\lambda(m^i) \tilde{f}_\lambda(\bar{m}^j), \tag{2.2}$$

as a sum of products of holomorphic functions f_λ and anti-holomorphic ones $\tilde{f}_\lambda$. If this is so, the $\{f_\lambda(m^i)\}$ span a finite-dimensional vector space $\mathscr{H}_\Sigma$ canonically associated with Σ. Σ may now be regarded purely as an oriented, topological surface; the moduli have been buried as arguments of the $f_\lambda(m^i)$, and forgetting the origin of the f_λs, we now merely think of them as a basis of a vector space $\mathscr{H}_\Sigma$.

Friedan & Shenker (1987) urged us to consider the behaviour of the $\mathscr{H}_\Sigma$ as Σ degenerates. For instance, we consider the degeneration of a surface Σ of genus g to surfaces Σ_1, Σ_2 of genus g_1 and g_2 (with $g_1 + g_2 = g$). As indicated in figure 1, it is natural to think of such a process as a time-dependent process in which a surface of genus $g_1 + g_2$ enters in the past and two surfaces of genus g_1 and g_2 emerge in the future. Such a space-time history naturally gives a three-manifold. To begin with, it appears that on this three-manifold there is a preferred 'time' coordinate, as it really represents a one-parameter family of two-dimensional configurations. I claim, however, that this is an illusion, and that one can consider the picture sketched in figure 1 to have full three-dimensional symmetry.

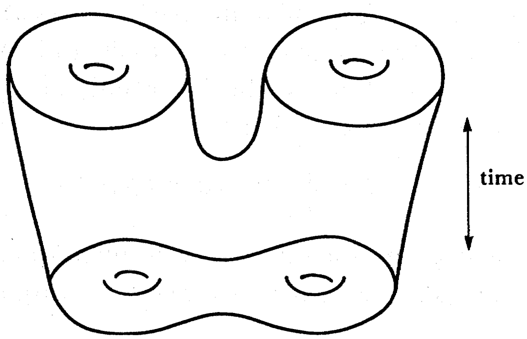

FIGURE 1. A Riemann surface of genus two breaking into two surfaces of genus one.

To further elaborate this claim, recall that in such theories the *primary fields* $\Phi_i(z, \bar{z})$ have decompositions

$$\Phi_i(z, \bar{z}) = \varphi_i(z)\, \tilde{\varphi}_i(\bar{z}), \tag{2.3}$$

where the $\varphi_i(z)$ are holomorphic but not quite local. In fact, the n-point correlation function,

$$G_{i_1, \ldots, i_n}(z_{i_1}, \ldots, z_{i_n}) = \langle \varphi_{i_1}(z_1) \ldots \varphi_{i_n}(z_n) \rangle, \tag{2.4}$$

is a multivalued function with, say, k branches, which we will call $G^\sigma_{i_1, \ldots, i_n}, \sigma = 1, \ldots, k$.

Forgetting about the origin of the G^{σ}s and just thinking of them as a basis of a vector space, we get a vector space $\mathscr{H}_{\Sigma, i_1, \ldots, i_n}$ that depends on the surface Σ as a topological surface with n points labelled as $i_1, \ldots, i_n$. The moduli z^k are irrelevant; they have disappeared as arguments of the G^{σ}.

When studying a multivalued function, such as the correlation function (see equation (2.4)), it is important to study the monodromies, that is, the linear transformations of the branches G^{σ} that occur when the points z_i loop around one another. It is convenient to think of the z_i as n points on the Riemann sphere (or complex plane C), that loop around one another in the course of time, forming a braid (figure 2*a*) in three-space. In figure 2*a*, only the topology of the braid is relevant. Thus one may make arbitrary time-dependent diffeomorphisms of the plane C in which the points are moving in figure 2*a*. I claim, though, that there is a higher, three-dimensional symmetry in figure 2*a*. In fact, if one glues together the top and bottom of the braid under discussion, one forms (see figure 2*b*) a knot. It turns out that one should think of this knot as an intrinsically three-dimensional object; that is, we now allow full three-dimensional diffeomorphisms.

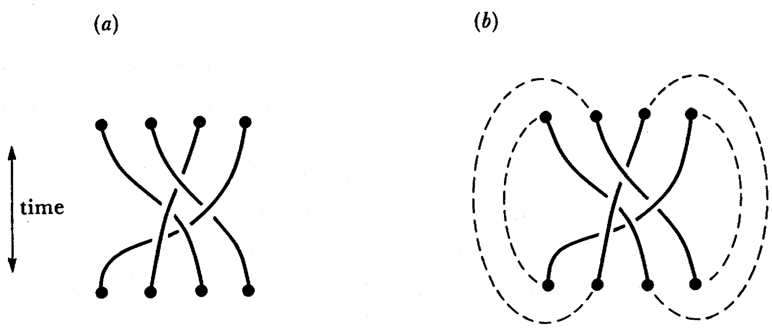

FIGURE 2. A braid (*a*) whose ends can be joined to make a knot (*b*).

To justify these assertions, one must obtain the vector spaces $\mathscr{H}_{\Sigma}$ and $\mathscr{H}_{\Sigma, i_1, \ldots, i_n}$ from a *three-dimensional viewpoint*. We have been urged (Atiyah 1988) to look to quantum field theory to provide this framework, and for physicists this is certainly natural.

Of course, a three-dimensional lagrangian,

$$\mathscr{L} = \int_M \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t\, W(\phi_i, \partial\phi_j) \tag{2.5}$$

(M denotes space-time and ϕ_i are some fields), will, upon quantization, associate a Hilbert space $\mathscr{H}_{\Sigma}$ to every Riemann surface Σ. If $\mathscr{L}$ is generally covariant, $\mathscr{H}_{\Sigma}$ will depend on Σ only as a *topological* surface.

So to understand the three-dimensional symmetry of two-dimensional conformal field theory, we look at generally covariant three-dimensional quantum field theories. General covariance does not allow us to introduce an *a priori* metric on M. If there is a metric at all, it must be a dynamical variable. But if so, the quantum wave-functions will depend upon the metric, and the quantum Hilbert spaces will be infinite dimensional. As we want finite-dimensional spaces (to agree with rational conformal field theory), we must avoid this, so we must consider three-dimensional theories in which general covariance is achieved with *no metric at all*. Thus we must relate two-dimensional conformal field theory to the *unbroken phase* of three-dimensional gravity, in which the metric is irrelevant.

The basic example seems to be Chern–Simons gauge theory. So (Witten 1989*b*) we pick a compact gauge group G and a positive integer k. We introduce a gauge field $A_i^a(x)$ and write the lagrangian

$$\mathscr{L} = \frac{k}{4\pi}\int_M \mathrm{Tr}\,(A \wedge \mathrm{d}A + \tfrac{2}{3}A \wedge A \wedge A). \tag{2.6}$$

No metric is needed, so this is generally covariant.

The reason that the quantum field theory with this lagrangian is exactly soluble is that this theory is trivial locally. For instance, at the classical level the Euler–Lagrange equation for this theory is $F_{ij}^a = 0$, and implies that A_i can be gauged away locally. Thus as in twistor theory, all information is to be encoded globally.

Quantum mechanically, to construct the Hilbert space $\mathscr{H}_\Sigma$ associated with a riemannian surface Σ, we pick the gauge $A_0 = 0$. The Gauss law constraint $0 = \delta\mathscr{L}/\delta A_0^a = F_{12}^a$ then tells us to consider only flat corrections. After dividing by gauge transformations, the phase space that must be quantized is what I shall call $\mathscr{M}_\Sigma$, the moduli space of flat G corrections on Σ, up to gauge equivalence. $\mathscr{M}_\Sigma$ is a compact symplectic manifold, and its quantization gives the mysterious Hilbert spaces $\mathscr{H}_\Sigma$ of G *current algebra at level k*. If one repeats this in the presence of static charges in representations $R_{i_1}, \ldots, R_{i_n}$ of G, one gets more general Hilbert spaces $\mathscr{H}_{\Sigma, i_1, \ldots, i_n}$ of multivalued correlation functions (see equation (2.4)) with Φ_i being a current algebra primary field in the representation R_i. In turn, the latter spaces are essentially the Jones braid representations (Jones 1985) that are associated with the celebrated Jones polynomial of knot theory; the relation of these spaces to conformal field theory was first perceived by Tsuchiya & Kanie (1988).

One might view these mysterious Hilbert spaces $\mathscr{H}_\Sigma$ as rather esoteric objects, but in fact, the whole current algebra theory in two dimensions can be reconstructed from the topological theory in three dimensions. As explained in the last few pages of (Witten 1989), this is done by quantizing the Chern–Simons theory on *Riemann surfaces with boundary*. For instance, we take Σ to be a disk D. Taking M to be $D \times R^1$, with R^1 denoting 'time', the lagrangian (see equation (2.6)) is only gauge invariant under the group G_1 consisting of gauge transformations that are 1 on the boundary of D. Quantizing in $A_0 = 0$ gauge, the Gauss law constraint still implies $F_{12}^a = 0$, so (as $D \times R^1$ is simply connected) one has

$$A_i(x,y) = \partial_i U \cdot U^{-1}, \tag{2.7}$$

where

$$U(x,y) \approx U(x,y) \cdot V \tag{2.8}$$

for any constant group element V (because $U \rightarrow UV$ leaves A unchanged in (equation (2.7)). Up to gauge transformation by G_1, only the value of U on the boundary of D is relevant. The boundary of D is a circle S^1, which we may parametrize by an angle θ, $0 \leqslant \theta \leqslant 2\pi$. So $U(\theta)$ defines a point in the loop space $\mathscr{L}$G of G; more exactly, in view of the equivalence of equation (2.8), we should think of $U(\theta)$ as a point in the homogeneous space $\mathscr{L}$G/G. Thus, it is $\mathscr{L}$G/G that is to be quantized. According to Segal (1981), quantization of $\mathscr{L}$G/G gives the irreducible (vacuum) representation of G current algebra at level k.

More generally, by quantizing on an n-holed surface of genus g (figure 3) one can study the n-point functions in genus g from the three-dimensional viewpoint.

It is not difficult to show abstractly that any rational conformal field theory gives rise automatically to a three-dimensional generally covariant theory. One approach to proving this

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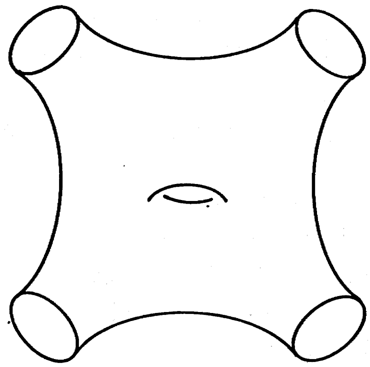

FIGURE 3. A surface of genus one with four holes.

(suggested to me by W. Thurston) involves decomposing a three manifold via a Morse function, by using the behaviour of the $\mathscr{H}_\Sigma$ under degeneration of Σ to define transition amplitudes in pictures such as that of figure 2, and then verifying that the choice of Morse function does not matter. Details will be described elsewhere.

More concretely, though, Moore & Seiberg (1989) have recently shown that G/H models may also be interpreted as Chern–Simons theories of an appropriate group (essentially G × H). It is easy to see that this is also true of rational orbifolds, and of certain generalizations of G/H models that will be described elsewhere. So it is possible to conjecture that arbitrary rational conformal field theories in two dimensions can be derived from Chern–Simons theories in three dimensions.

Actually one can make a stronger conjecture along these lines. Apart from the models that are related to two-dimensional conformal field theories, the only three-dimensional generally covariant theories I know of that at first sight appear not to be Chern–Simons theories are general relativity and the three-dimensional reduction of Donaldson theory. But both three-dimensional gravity (Achúcarro & Townsend 1986; Witten 1989*a*) and the three-dimensional reduction of Donaldson theory (Witten 1989*c*) have turned out to be Chern–Simons theories. So one may conjecture that every three-dimensional generally covariant theory is actually a Chern–Simons theory of some group or supergroup, not necessarily connected or simply connected. Perhaps this conjecture should be stated only in the context of the unbroken phase of general relativity.

Integrable lattice models in $1+1$ dimensions are also intimately connected with three-dimensional quantum field theory, as is apparent from the literature of recent years on knot theory and statistical mechanics, along with the above observations. I will try elsewhere (Witten 1989*d*) to elucidate at least part of this connection from a physical point of view.

In conclusion, if one considers rational conformal field theories and their cousins, integrable systems, in two dimensions, one sees that these are extremely rich systems. There are an incredible variety of extremely rich facts, related to each other in an incredible diversity of ways. To bring order to this chaos looks hopeless. But by stepping out of flatland and looking at things from the vantage point of three dimensions, one can find a more powerful viewpoint, where rational and integrable systems can be derived from a subtler and more incisive starting point. This step out of flatland, to a higher vantage point from which wider symmetry can be seen, is temptingly akin to what we need in string theory. What is more, we have been urged

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(Atiyah 1988) to take yet another step to four dimensions, the most physical dimension, the richest dimension for geometry, and the critical dimension for quantum field theory.

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References

Achúcarro, A. & Townsend, P. 1986 *Phys. Lett.* B **180**, 89.
Atick, J. & Witten, E. 1988 The Hagedorn transition and the number of degrees of freedom in string theory. *Nucl. Phys.* B **310**, 291.
Atiyah, M. F. 1988 New invariants of three and four dimensional manifolds. In *The mathematical heritage of Hermann Weyl*, *Proc. Symp. pure Math.* **48** (ed. R. O. Wells, Jr). Providence: AMS.
Friedan, D. & Shenker, S. 1987 *Nucl. Phys.* B **281**, 509.
Isenberg, J., Yasskin, P. & Green, P. 1978 *Phys. Lett.* B **78**, 462.
Jones, V. F. R. 1985 A polynomial invariant for links via subfactors. *Bull. AMS* **12**, 103.
Lerner, M. & Sommers, D. (ed.) 1978 *Complex manifold techniques in theoretical physics*. Pitman.
Moore, G. & Seiberg, N. 1989 Taming the conformal zoo. IAS preprint HEP-89/6.
Polyakov, A. M. 1978 *Phys. Lett.* B **72**, 477.
Segal, G. 1981 Unitary representations of some infinite dimensional groups. *Communs math. Phys.* **80**, 301.
Susskind, L. 1979 *Phys. Rev.* D **20**, 2610.
Tsuchiya, A. & Kanie, Y. 1988 In *Conformal field theory and solvable lattice models*. *Adv. Studies pure Math.* **16**, 297.
Witten, E. 1978 An interpretation of classical Yang–Mills theory. *Phys. Lett.* B **77**, 394.
Witten, E. 1989 *a* 2+1 Dimensional gravity as an exactly soluble system. *Nucl. Phys.* B **311**, 46.
Witten, E. 1989 *b* Quantum field theory and the Jones polynomial. *Communs math. Phys.* **121**, 351.
Witten, E. 1989 *c* Topology changing amplitudes in 2+1 dimensional gravity. *Nucl. Phys.* B. (In the press.)
Witten, E. 1989 *d* Gauge theories and integrable lattice models. *Nucl. Phys.* B. (In the press.)

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